FACIAL STRUCTURE OF THE CONE OF NONNEGATIVE TERNARY QUARTICS

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Abstract. In this work we will discuss the facial structure of the cone of nonnegative ternary quartics with real coefficients. We will establish an equivalence relation on the set of all faces, which preserves certain properties like dimension or the number of common zeros. Moreover we will give a complete list of equivalence classes and we will discuss their dimension and inclusions between them. Eventually we will be able to present a complete lattice of all faces of this particular cone.

1. Introduction

We write $\mathbb{P}^k = \mathbb{P}(\mathbb{C}^{k+1})$ for the k–dimensional projective space over $\mathbb{C}$. Let $\mathbb{P}^k(\mathbb{R})$ denote the real points of $\mathbb{P}^k$. Let $\mathrm{H}_k \subseteq \mathbb{R}[x,y,z]$ be the vector space of real polynomials in three variables homogeneous of degree k. For $I \subseteq \mathrm{H}_k$ an arbitrary subset we write $\mathcal{Z}(I) := \{x \in \mathbb{P}^2(\mathbb{R})\colon\ f(x) = 0,\ \forall f \in I\}$ for the set of all *real* zeros of I. We set

$$\mathrm{P}_{3,4} = \{p \in \mathrm{H}_4\colon\ p(x) \geq 0,\ \text{for all}\ \ x \in \mathbb{P}^n(\mathbb{R})\}$$

and call $P_{3,4}$ the *cone of nonnegative ternary quartics*. Evidently $\mathrm{P}_{3,4}$ is closed under addition and multiplication with nonnegative reals. Hilbert ([Hi]) proved in 1888 that every polynomial in $\mathrm{P}_{3,4}$ is a sum of squares of quadratic forms. Yet still little is known about the facial structure of cones of nonnegative forms. In this work we will explore the facial structure of $\mathrm{P}_{3,4}$ and show that $\mathrm{P}_{3,4}$ has up to the natural $\mathrm{PGL}_3(\mathbb{R})$–action only finitely many faces. We will give a complete list of these, and discuss issues as exposednes and inclusions between faces.

2. Basic concepts

2.1. **Convex Cones.**

Definition 2.1. A set $C \subseteq \mathbb{R}^k$ is called a *convex cone*, if it is closed under addition and multiplication with nonnegative reals. A subcone $F \subseteq C$ is called

face of C, if for any $a, b \in C$, whenever $a + b \in F$, we must have $a, b \in F$. For a face $F \subseteq C$ we define the dimension of F by

$$\dim F := \dim \operatorname{span}(F).$$

An element $u \neq 0$ contained in a one-dimensional face is called extremal element, for $C = \mathrm{P}_{3,4}$ it is also called extremal form. Let V be a vector subspace of $\mathbb{R}^k$. We call F *full* in V, if $V = \operatorname{span}(F)$, or equivalently if $F = C \cap V$ and there exists $f \in F$ such that for all $g \in V$ there exists $\epsilon > 0$ with $f + \epsilon g \in C$. To every $f \in C$ we attach the set

$$F_f := \{g \in C \colon\ f - \epsilon g \in C \text{ for some } \epsilon > 0\}.$$

A face $F \subseteq C$ is called exposed if there exists a linear functional $L : \mathbb{R}^k \longrightarrow \mathbb{R}$, such that L is nonnegative on C and $F = \ker(L) \cap C$. In this case we say that L defines F.

Remark 2.2. It is easy to check that F_f is a face of C. It is the unique face containing f as an inner point. Since every face of C has an inner point, every face $F \subseteq C$ can be written as F_f for a suitable $f \in C$. By construction $g \in C$ is an inner point of F_f if and only if $F_g = F_f$. If $C = \mathrm{P}_{3,4}$ then $f \in F_g$ implies $\mathcal{Z}(g) \subseteq \mathcal{Z}(f)$. In particular $F_f = F_g$ only if $\mathcal{Z}(f) = \mathcal{Z}(g)$.

2.2. **Infinitely near points.**

Let $f \in \mathrm{H}_4$, $p \in \mathbb{P}^2$ with $\operatorname{ord}_p(f) \geq 2$. We choose affine coordinates x, y in p (i.e. p becomes the origin). We choose projective coordinates z, w of $\mathbb{P}^1$ and write $X := \mathcal{V}(xw - yz) \subset \mathbb{A}^2 \times \mathbb{P}^1$ for the blowup of $\mathbb{P}^2$ in p with the corresponding projection $\pi \colon X \longrightarrow \mathbb{A}^2$. We denote the ideal of the exceptional line by $\mathfrak{q} := (x, y) \subset \mathbb{R}[x, y, z, w]$. For f as above we define the ideal

$$T := \left((\pi^*(f), xw - yz) : \mathfrak{q}^2\right),$$

which is homogeneous in z, w and defines a curve in X.

Remark 2.3. If we have $\operatorname{ord}_p(f) = 2$ in above construction then $\mathcal{V}(T)$ is the strict transform of f. If $\operatorname{ord}_p(f) > 2$ then $\mathcal{V}(T)$ is the union of the strict transform and the exceptional line.

Definition 2.4. A (first order) real infinitely near point of f in p is a real point on $\mathcal{V}(T + q) \subseteq \mathbb{P}^1$. We denote the set of all first order real infinitely near points of f in p by $\operatorname{inp}_f(p)$.

Definition 2.5. For a line l through p we write $l_p \in \mathbb{P}^1$ for the intersection of the strict transform of l in p with the exceptional line $\mathcal{V}(\mathfrak{q})$. Each point on the exceptional line can be written as l_p for a suitable line l.

Definition 2.6. For a fixed open embedding $\phi\colon A \hookrightarrow X$, i.e. affine coordinates on X, we denote by $\widetilde{f}$ the generator of $\phi^*(T)$.

Remark 2.7. In this paper we will always choose the the open embedding

$$\begin{aligned} \mathbb{A}^2 &\longrightarrow X, \\ (x,y) &\longmapsto (x, xy; 1:y). \end{aligned}$$

Under this choice we have $\widetilde{f}(x,y) = \frac{f(x,xy)}{x^2}$.

3. Constructing faces

Definition 3.1.

(i) For a form $q \in \mathrm{H}_k$ and $p \in \mathbb{P}^2(\mathbb{R})$ with $q(p) = 0$ we set

$$\mathbb{T}_p(q) := \{x \in \mathbb{P}^2\colon\ \langle x, \nabla q(p)\rangle = 0\};$$

the *tangent space of q in p.*

(ii) For a subset $J \subseteq H_k$ we define the vector space

$$J^{[2]} := \operatorname{span}\left(\{fg\colon\ f,g \in J\}\right) \subseteq H_{2k}.$$

(iii) For a face $F \subseteq \mathrm{P}_{3,4}$ we set

$$J_F := \{q \in \mathrm{H}_2\colon\ q^2 \in F\} \subseteq \mathrm{H}_2\,.$$

Lemma 3.2. *For an arbitrary face $F \subseteq \mathrm{P}_{3,4}$ the set J_F is a vector space.*

Proof. Obviously J_F is closed under scalar multiplication. For $q_1, q_2 \in J_F$ we have $2(q_1^2 + q_2^2) = (q_1+q_2)^2 + (q_1 - q_2)^2 \in F$, which proves that $q_1 \pm q_2 \in J_F$. $\square$

Let $p \in \mathbb{P}^2(\mathbb{R})$ and l a real line containing p.

Definition 3.3. For p, l as above we define following sets

$$\begin{aligned} F_p &:= \{f \in \mathrm{P}_{3,4}\colon\ f(p) = 0\} \\ I_p &:= \{f \in \mathrm{H}_4\colon\ \operatorname{ord}_p(f) \geq 2\} \\ F_{(p,l)} &:= \left\{f \in \mathrm{P}_{3,4}\colon\ f(p) = 0,\ \widetilde{f}(l_p) = 0\right\} \\ I_{(p,l)} &:= \left\{f \in \mathrm{H}_4\colon\ \operatorname{ord}_p f \geq 2,\ \operatorname{ord}_{l_p}(\widetilde{f}) \geq 2\right\}, \end{aligned}$$

where all occurring blow-ups are taken in p.

Lemma 3.4. *Let $p \in \mathbb{P}^2(\mathbb{R})$ and l a real line containing p. Then*

(i) $F_p, F_{(p,l)} \subseteq \mathrm{P}_{3,4}$ *are faces.*
(ii) $I_p \cap \mathrm{P}_{3,4} = F_p$ *and* $I_{(p,l)} \cap \mathrm{P}_{3,4} = F_{(p,l)}$.
(iii) $J_F^{[2]} = \mathrm{span}(F)$.
(iv) *The face F_p is exposed.*

Proof.

(i) Let $f, g \in F_{(p,l)}$ and $\lambda \geq 0$. Obviously we have $(f+g)(p) = 0$, $(\lambda f)(p) = 0$, as well as $\widetilde{(f+g)}(l_p) = \widetilde{f}(l_p) + \widetilde{g}(l_p) = 0$ and $\widetilde{(\lambda f)}(l_p) = \lambda \widetilde{f}(l_p) = 0$. This shows, that F_p and $F_{(p,l)}$ are convex subcones of $\mathrm{P}_{3,4}$. Now suppose $f, g \in \mathrm{P}_{3,4}$ such that $f + g \in F_{(p,l)}$. Since f, g are nonnegative and $(f+g)(p) = 0$, we must have $f(p) = g(p) = 0$. Also $\widetilde{f}, \widetilde{g}$ are nonnegative polynomials, which shows, that $(\widetilde{f} + \widetilde{g})(l_p) = 0$ implies $\widetilde{f}(l_p) = 0$ and $\widetilde{g}(l_p) = 0$.

(ii) Suppose $f \in F_{(p,l)}$. It suffices to show that $f \in I_{(p,l)}$. Since f is nonnegative, p is a local minimum of f and therefore $\mathrm{ord}_p(f) \geq 2$. The same argument shows that $\mathrm{ord}_{l_p}(\widetilde{f})$ is even and positive and therefore at least 2.

(iii) Let $f, g \in J_F$. Then $4fg = (f+g)^2 - (f-g)^2 \in \mathrm{span}(F)$ and therefore $J_F^{[2]} \subseteq \mathrm{span}(F)$. Now suppose $f \in \mathrm{span}(F)$. Then $f = f_1 - f_2$ with $f_1, f_2 \in F$. Since every nonnegative ternary quartic is a sum of squares, there exist $q_1, \ldots, q_r \in H_2$ such that $f = q_1^2 + \cdots + q_t^2 - q_{t+1}^2 - \cdots - q_r^2$. Since F is a face, we have $q_1^2, \ldots, q_r^2 \in F$ and therefore $q_1, \ldots, q_r \in J_F$.

(iv) Evaluation in p is a linear functional $L_p \colon \mathrm{H}_{3,4} \longrightarrow \mathbb{R}$, which is nonnegative on $\mathrm{P}_{3,4}$. Obviously we have $\ker(L_p) \cap \mathrm{P}_{3,4} = F_p$.

$\square$

Convention 3.5. We will write shortly J_p and $J_{(p,l)}$ instead of J_{F_p} and $J_{F_{(p,l)}}$ respectively. Similarly we write J_f instead J_{F_f} for $f \in \mathrm{P}_{3,4}$.

Lemma 3.6. *Let p, l be as above*

(i) $J_p = \{q \in \mathrm{H}_2 \colon\ q(p) = 0\}$,
(ii) $J_{(p,l)} = \{q \in \mathrm{H}_2 \colon\ q(p) = 0,\ l \subseteq \mathbb{T}_p(q)\}$,

Proof. While the first assertion is obvious, the second one needs some explanation. We choose affine coordinates in p such that l becomes the line $y = 0$. For

$q \in \mathrm{H}_2$ with $q(p) = 0$ and $\nabla q(p) = (a, b)^t$ we have

$$\begin{aligned} q^2 \in F_{(p,l)} &\iff q(p) = 0 \text{ and } \widetilde{q^2}(0,0) = 0 \\ &\iff q(p) = 0 \text{ and } (a + by)^2_{|y=0} = 0 \\ &\iff q(p) = 0 \text{ and } a = 0 \\ &\iff q(p) = 0 \text{ and } \nabla q(p) \perp (1,0) \\ &\iff q(p) = 0 \text{ and } l \subseteq \mathbb{T}_p(q). \end{aligned}$$

$\square$

Definition 3.7. Let $\mathsf{S} = \{s_1, \ldots, s_n, (p_1, l_1), \ldots, (p_m, l_m)\}$ where $s_1, \ldots, s_n, p_1, \ldots, p_m \in \mathbb{P}^2(\mathbb{R})$ are pairwise different and $l_1, \ldots, l_m$ are real lines such that $p_i \in l_i$ for $i = 1, \ldots, m$. We attach following sets to S.

$$F_{\mathsf{S}} := \bigcap_{i=1}^{n} F_{s_i} \cap \bigcap_{j=1}^{m} F_{(p_j,l_j)}$$

$$I_{\mathsf{S}} := \bigcap_{i=1}^{n} I_{s_i} \cap \bigcap_{j=1}^{m} I_{(p_j,l_j)}$$

Corollary 3.8. *For S as above we have*

(i) $F_{\mathsf{S}} \subseteq \mathrm{P}_{3,4}$ *is a face.*
(ii) $I_{\mathsf{S}} \cap \mathrm{P}_{3,4} = F_{\mathsf{S}}$

Similarly as in Convention 3.5 we write shortly J_{S} instead of $J_{F_{\mathsf{S}}}$.

Remark 3.9. The idea to prescribe certain configurations of zeros is due to Blekherman [Bl]. He used this approach to give an estimate of the dimension of certain faces of an *arbitrary* cone of nonnegative forms. Moreover he was able to point out dimensional differences between faces of nonnegative forms and corresponding faces of the cone of sum of squares.

4. Fullness

In the last section we have seen, that $\mathrm{span}(F_{\mathsf{S}}) \subseteq I_{\mathsf{S}}$. In general this inclusion is strict as the following example shows. Since $\dim(I_{\mathsf{S}})$ can be computed explicitly, we are interested in the case for which equality holds.

Example 4.1. Let $\mathsf{S} = \{(1\colon 0\colon 0), (0\colon 1\colon 0), (1\colon 1\colon 0)\}$. Each form $f \in F_{\mathsf{S}}$ meets the line $\mathcal{V}(z)$ in three double points. Therefore Bezout's theorem implies that f can be written as $f = zq$ with a cubic q. Since f is positive, the gradient of

f must vanish on all real points of the line $\mathcal{V}(z)$. Thus q vanishes on $\mathcal{V}(z)$ and hence $z \mid q$. Thus each form in $F_{\mathcal{S}}$ is divisible by z^2. On the other hand we have $g := xy^2z - xyz^2 \in I_{\mathcal{S}}$ and therefore $\operatorname{span}(F_{\mathcal{S}}) \subsetneq I_{\mathcal{S}}$.

According to definition 2.1 we have $\operatorname{span}(F_{\mathcal{S}}) = I_{\mathcal{S}}$ if and only if there exists $f \in F_{\mathcal{S}}$ such that for all $g \in I_{\mathcal{S}}$ there exists an $\epsilon > 0$ such that $f + \epsilon g \in \mathrm{P}_{3,4}$. We will now find a criterion on $\mathcal{S}$, whether such an f can be constructed.

Definition 4.2. Let $f \in \mathrm{H}_4$ and $p \in \mathbb{P}^2(\mathbb{R})$ such that $\operatorname{ord}_p(f) \geq 2$. We choose affine coordinates in p and define the discriminant $\mathrm{D}_f(y)$ of f in p as the polynomial $\mathrm{D}_f(y) = (f_3^2 - 4f_2f_4)(1, y)$ where f_2, f_3, f_4 are the homogeneous components of f. Using $\widetilde{f}$ according to 2.7 we conclude that $\mathrm{D}_f(y)$ is the discriminant with respect to x of the quadratic polynomial $\widetilde{f}(y) \in \mathbb{R}[x]$.

Lemma 4.3. *Let $p \in \mathbb{P}^2(\mathbb{R})$ and l a real line through p. Suppose $f \in F_{(p,l)}$ and $g \in I_{(p,l)}$. If*

(i) $\operatorname{ord}_p(f) = 2$
(ii) $(\mathrm{D}_f)''(0) \neq 0$

then there exists a neighborhood $U \subseteq \mathbb{P}^2(\mathbb{R})$ of p and $\epsilon > 0$ such that $f + \epsilon g$ is nonnegative on U.

Proof. We choose affine coordinates in p so that l is the line $y = 0$. According to remark 2.7 we have $l_p = 0$. We write $f_k = \sum_{i=0}^{k} a_{i,k-i} x^i y^{k-i}$ and $g_k = \sum_{i=0}^{k} b_{i,k-i} x^i y^{k-i}$ for the homogeneous components of f and g respectively. Since $f, g \in I_{(p,l)}$ we conclude

$$a_{20} = a_{11} = a_{30} = 0,$$
$$b_{20} = b_{11} = b_{30} = 0.$$

Thus $y \mid f_3, g_3$ and $y^2 \mid f_2, g_2$. Since $\widetilde{f}$ is nonnegative, $\mathrm{D}_f \leq 0$ and $y = 0$ is a locale maximum of D_f. Fix $t > 0$, then we have

$$\mathrm{D}_{f+tg}(y) = \mathrm{D}_f(y) + t^2 P(y) + tQ(y)$$

for suitable $P, Q \in \mathbb{R}[y]$ with $y^2 \mid P(y), Q(y)$. Considering $(\mathrm{D}_f)''(0) \neq 0$ we find $\delta, \epsilon_0 > 0$ such that $\mathrm{D}_{f+tg}(y) \leq 0$ as long as $|y| \leq \delta$ and $0 < t < \epsilon_0$. Since f is nonnegative and $\operatorname{ord}_p(f) = 2$ we must have $a_{02} > 0$. We choose $0 < \epsilon < \epsilon_0$ such that $a_{02} + \epsilon b_{02} =: c$ is positive, which implies that $f_2 + \epsilon g_2 = cy^2$ is a nonnegative form. We set $h := (f + \epsilon g)$. Let $v := (u, w) \in \mathbb{R}^2$ with $|v| = 1$. We have

$$h(\lambda v) = c\lambda^2 w^2 + \lambda^3 R(v, \lambda)$$

for a suitable $R \in \mathbb{R}[v, \lambda]$. Hence there exists $\mu > 0$ such that $h(\lambda v) \geq 0$ if $\lambda \leq \mu$ and $\left|\frac{w}{u}\right| \geq \delta$. Since $\mathrm{D}_h(y) \leq 0$ for $y \leq \delta$ we conclude $h(x, y) \geq 0$ if $\left|\frac{y}{x}\right| \leq \delta$. Thus h is nonnegative on $B_\mu(0)$. □

Theorem 4.4. *Let $\mathcal{S} = \{s_1, \dots, s_n, (p_1, l_1), \dots, (p_m, l_m)\}$ be as above. If there exists $f \in F_{\mathcal{S}}$ such that*

(i) $\mathcal{Z}(f) = \{s_1, \dots, s_n, p_1, \dots, p_m\}$,
(ii) $\mathrm{inp}_f(s_i) = \emptyset$,
(iii) $\mathrm{ord}_{p_j}(f) = 2$,
(iv) $(\mathrm{D}_f)''(0) \neq 0$ *in* p_j,

for all $i = 1, \dots, n$ and all $j = 1, \dots, m$ then $\mathrm{span}(F_{\mathcal{S}}) = I_{\mathcal{S}}$ *and in particular* $\dim I_{\mathcal{S}} = \dim F_{\mathcal{S}}$.

Proof. Let $g \in I_{\mathcal{S}}$ and $t \in \mathbb{P}^2(\mathbb{R})$. If $f(t) \neq 0$ there exists $\epsilon_t > 0$ and an open $U_t \subseteq \mathbb{P}^2(\mathbb{R})$ containing t such that $f + \epsilon_t g$ is nonnegative on U_t. If $t \in \{p_1, \dots, p_m\}$ the preceding lemma implies the existence of an $\epsilon_t > 0$ and an open $U_t \subseteq \mathbb{P}^2$ containing t such that $f + \epsilon t g$ is nonnegative on U_t. Suppose now $t \in \{s_i, \dots, s_n\}$. Since $\mathrm{inp}_f(t) = \emptyset$ the form f_2 (in t) is positive definite. Hence there exists ϵ_t and U_t like above, such that $f + \epsilon_t g$ is nonnegative on U_t. Since $\mathbb{P}^2(\mathbb{R})$ is compact we can cover $\mathbb{P}^2(\mathbb{R})$ by finitely many of these U_t. Choosing ϵ minimal among all occurring ϵ_t shows that $f + \epsilon g$ is nonnegative on $\mathbb{P}^2(\mathbb{R})$ and thus $g \in \mathrm{span}(F_{\mathcal{S}})$. □

Since the above condition on the discriminant of f is rather bulky, we provide a more algebraic respectively algorithmic version of above theorem in the following section.

5. Algebraic conditions

Let $\mathcal{S} = \{s_1, \dots, s_n, (p_1, l_1), \dots, (p_m, l_m)\}$ be defined as in the preceding section. Since $J_{\mathcal{S}}^{[2]} = \mathrm{span}(F_{\mathcal{S}})$ we can work rather with $J_{\mathcal{S}}$ than with $F_{\mathcal{S}}$.

Definition 5.1. For a vector space $J \subseteq \mathrm{H}_2$ and $p \in \mathbb{P}^2(\mathbb{R})$ we define

$$G_J(p) = \{\nabla q(p) \colon \; q \in J\} \subseteq \mathbb{P}^2(\mathbb{R})$$
$$E_J(p) = \{q \in J \colon \; \mathrm{ord}_p(q) \geq 2\}$$

Theorem 5.2. *Let $J \subseteq J_{\mathfrak{S}}$. If*

(i) $\mathcal{Z}(J) = \{s_1, \dots, s_n, p_1, \dots, p_m\}$
(ii) $\dim G_J(s_i) = 1$
(iii) $G_J(p_j) \neq \emptyset$
(iv) $\mathcal{Z}(E_J(p_j)) \cap l_j = \{p_j\}$

for all $i = 1, \dots, n$ and all $j = 1, \dots, m$, then $F_{\mathfrak{S}}$ is full in $I_{\mathfrak{S}}$. Moreover for any basis $q_1, \dots, q_k$ of J the form $q_1^2 + \dots + q_k^2$ is an inner form of $F_{\mathfrak{S}}$.

Proof. Let $q_1, \dots, q_k$ be a basis of J and set $f := q_1^2 + \dots + q_k^2$. We show that f satisfies all conditions of Theorem 4.4. The first condition implies $\mathcal{Z}(f) = \{s_1, \dots, s_n, p_1, \dots, p_m\}$. Fix $s \in \{s_1, \dots, s_n\}$ and choose affine coordinates in s. The second condition implies $k \geq 2$ and we can assume that $\nabla q_1(s) := (a_1, b_1)^t$ and $\nabla q_2(s) := (a_2, b_2)^t$ are linearly independent. We may assume that $f = q_1^2 + q_2^2$. We show that $\operatorname{inp}_f(s) = \emptyset$. We have

$$\widetilde{g}(0, y) = (a_1^2 + a_2^2) + 2(a_1 b_1 + a_2 b_2)y + (b_1^2 + b_2^2)y^2.$$

This polynomial has discriminant

$$4(a_1 b_1 + a_2 b_2)^2 - 4(a_1^2 + a_2^2)(b_1^2 + b_2^2) = -(a_1 b_2 - a_2 b_1)^2 = -\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 < 0.$$

Moreover $(b_1^2 + b_2^2) \neq 0$ and thus $\operatorname{inp}_f(s) = \emptyset$. Fix $(p, l) \in \{(p_1, l_1) \dots, (p_m, l_m)\}$ and choose affine coordinates in p such that l is the line $\mathcal{V}(y)$. Since $\widetilde{f}(0, 0) = 0$ we must have $\nabla q_i(p) = (0, b_i)$ for $i = 1, \dots, k$ and suitable $b_i \in \mathbb{R}$. Condition (iii) implies that there exists $i \in \{1, \dots, k\}$ such that $b_i \neq 0$ and hence $\operatorname{ord}_p(f) = 2$. We write $q_i(x, y) = b_i y + c_i x^2 + d_i xy + e_i y^2$ with $c_i, d_i, e_i \in \mathbb{R}$ and $i = 1, \dots, k$. Condition (iv) implies that the vectors $b := (b_1, \dots, b_k)$ and $c := (c_i, \dots, c_k)$ are linearly independent. On the other hand we have

$$\mathrm{D}_f(y) = (f_3^2 - 4 f_2 f_4)(1, y)$$

and thus

$$\begin{aligned} (\mathrm{D}_f)''(0) &= 8\left(\sum_{i=1}^k b_i c_i\right)^2 - 8\left(\sum_{i=1}^k b_i^2\right)\left(\sum_{i=1}^k c_i^2\right) \\ &= \sum_{i,j=1}^k \left(b_i b_j c_i c_j - b_i^2 c_j^2\right) \end{aligned}$$

$$= \sum_{i,j=1}^{k} \left(b_i c_j (c_i b_j - b_i c_j)\right)$$
$$= \sum_{i<j} \left(c_i b_j - b_i c_j\right)^2 .$$

The last sum runs over all 2×2–Minors of the Matrix (b, c). Therefore $(\mathrm{D}_f)''(0) \neq 0$. This shows, that $\operatorname{span}(F_{\mathcal{S}}) = I_{\mathcal{S}}$ and f is an inner form of $F_{\mathcal{S}}$. □

Remark 5.3. Theorem 4.4 as well as theorem 5.2 can be generalized to quartics with an arbitrary finite number of variables.

6. Equivalence

Let $\mathcal{G} := \mathrm{PGL}_3(\mathbb{R})$ denote the real projective linear group on $\mathbb{P}^2$. If we identify positive multiples of forms in H_4 (resp. H_2) the group $\mathcal{G}$ operates on H_4 (resp. on H_2) by

$$(\sigma \cdot f)(x) := f(\sigma^{-1}(x)), \quad f \in \mathrm{H}_4 \ \ (\text{resp. } f \in \mathrm{H}_2), \sigma \in \mathcal{G}.$$

For $\mathcal{S} = \{s_1, \ldots, s_n, (p_1, l_1), \ldots, (p_m, l_m)\}$ as before and $\sigma \in \mathcal{G}$ we define

$$\sigma(\mathcal{S}) = \{\sigma(s_1), \ldots, \sigma(s_n), (\sigma(p_1), \sigma(l_1)), \ldots, (\sigma(p_m), \sigma(l_m)\}.$$

Lemma 6.1. *For $\mathcal{S}$ as above, $\sigma \in \mathcal{G}$ and $F \subseteq \mathrm{P}_{3,4}$ a face we have*

(i) $\sigma(F)$ is a face of $\mathrm{P}_{3,4}$ and $\dim(F) = \dim(\sigma(F))$.
(ii) $\sigma(F_{\mathcal{S}}) = F_{\sigma(\mathcal{S})}$, $\sigma(I_{\mathcal{S}}) = I_{\sigma(\mathcal{S})}$ and $\sigma(J_{\mathcal{S}}) = J_{\sigma(\mathcal{S})}$.
(iii) F exposed implies $\sigma(F)$ exposed.

Proof. The first assertion is obvious, since σ is linear and invertible and satisfies $\sigma(\mathrm{P}_{3,4}) = \mathrm{P}_{3,4}$. The second assertion is obvious if $m = 0$, thus it is enough to show the claim for $\mathcal{S} = \{(p, l)\}$. First we notice that $\operatorname{ord}_p(f) = \operatorname{ord}_{\sigma(p)}(\sigma(f))$ for all $f \in H_4$. We choose affine representatives of p and σ, which we will also denote by p and σ respectively. We write $p' = \sigma(p) \in \mathbb{R}^3$ and choose q such that $l = \operatorname{span}(p, q)$. For $q' := \sigma(p')$ we have $\sigma(l) = \operatorname{span}(p', q')$. Let further be $r \notin l$. Using r and $\sigma(r)$ respectively we construct affine coordinates x, y in p such that l becomes the line $\mathcal{V}(y)$ and affine coordinates x', y' in p' such that $\sigma(l)$ becomes the line $\mathcal{V}y'$ respectively. In this coordinates we compute f and

$\sigma(f)$ respectively and obtain

$$\begin{aligned}\tilde{f}(x,y) &:= f(p + x(p-q) + yr)\\ \widetilde{(\sigma f)}(x',y') &:= (\sigma f)(p' + x'(p'-q') + y'\sigma(r))\\ &= f(\sigma^{-1}(p' + x'(p'-q') + y'\sigma(r)))\\ &= f(p + x'(p-q) + y'r)\end{aligned}$$

which makes apparent, that $\mathrm{ord}_{l_p}(\widetilde{f}) = \mathrm{ord}_{\sigma(l)_{\sigma(p)}}(\widetilde{\sigma(f)})$.

Let L be a linear functional which defines F. If we fix an affine representative for σ we have a linear map $\varphi\colon \mathrm{H}_{3,4} \longrightarrow \mathrm{H}_{3,4}$ given by $f \mapsto \sigma f$. Then $L \circ \varphi$ is a linear functional, which is nonnegative on $\mathrm{P}_{3,4}$. For $f \in \mathrm{P}_{3,4}$ we have

$$f \in \ker(L \circ \varphi) \iff \sigma f \in \ker(L).$$

$\square$

Definition 6.2. Let $F_1, F_2 \subseteq \mathrm{P}_{3,4}$ be faces. We call F_1 equivalent to F_2 if there exists $\sigma \in \mathcal{G}$ such that $\sigma(F_1) = F_2$. In this case we write $F_1 \sim F_2$. Evidently $\sim$ is an equivalence relation. Denote $\mathcal{F}$ the quotient set. For $[F] \in \mathcal{F}$ we define $\dim([F]) = \dim(F)$, due to the preceding lemma this definition does not depend on the choice of a particular representative.

Henceforth we will restrict our considerations on equivalence classes of faces.

Definition 6.3. For $k \in \mathbb{N}$ we set

$$\mathbb{V}_k := \left\{(L_1, \ldots, L_k)\colon\ L_i \subset \mathbb{P}^2(\mathbb{R}) \text{ subspace},\ -1 \le \dim L_1 \le \ldots \le \dim L_k \le 2\right\}.$$

With the convention $g \cdot \emptyset = \emptyset$ for $g \in \mathcal{G}$ the group $\mathcal{G}$ operates on $\mathbb{V}_k$ entrywise.

The next lemma is a well-known fact from linear algebra.

Lemma 6.4. *Following sets form (complete) $\mathcal{G}$–orbits in $\mathbb{V}_4$.*

$\{(p_1, p_2, p_3, p_4)\colon\ p_i$ points, no 3 of them collinear $\}$

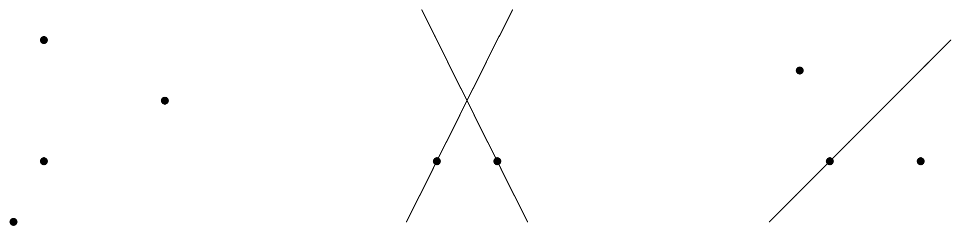
$\{(p_1, p_2, l_1, l_2)\colon\ p_i$ points, l_i lines, $p_i \in l_i,\ p_i \notin l_j,\ i \neq j\}$

$\{(p_1, p_2, p_3, l)\colon\ p_1, p_2, p_3$ non-collinear points, l a line, $p_1 \in l,\ p_2, p_3 \notin l\}$

In particular $\mathcal{G}$ operates transitively on each of these sets projected on a $\mathbb{V}_k$, with $1 \le k \le 4$.

Obviously $\mathbb{V}_4$ splits in far more orbits, but only the mentioned ones are of interest for us.

Remark 6.5. According to above orbits, it is useful to keep following pictures in mind,



which display the above configurations S, on which $\mathcal{G}$ operates transitively.

Lemma 6.6. *Let $f \in \mathrm{P}_{3,4}$.*

(i) Suppose f has three zeros on the real line l, then f is a product of l^2 and a nonnegative quadratic form.
(ii) Let $p, q \in \mathbb{P}^2(\mathbb{R})$ distinct and $l = p \vee q$. Suppose $f(p) = f(q) = 0$. If $l_p \in \operatorname{inp}_p(f)$ then f vanishes on the entire line l. In particular f is a product of l^2 and a nonnegative quadratic form.

Proof. Since $f \in \mathrm{P}_{3,4}$ we can write $f = g_1^2 + \cdots + g_k^2$ for suitable $g_i \in \mathrm{H}_2$. For the first statement we note, that $g_1, \ldots, g_k$ vanish in each of the three points on l. Then Bezout's Theorem implies that each g_i is divisible by l and the claim follows. For the second statement we note that $\nabla g_i(p)$ and ∇l are linearly dependent for all $i = 1, \ldots, k$. Therefore g_i and l have a at least a double intersection at p. Again Bezout's Theorem implies that each g_i is divisible by l. $\square$

Corollary 6.7. *Let $F \subseteq \mathrm{P}_{3,4}$ be a face, then exactly one of the following statements holds*

($\mathcal{A}$) *$\mathcal{Z}(F)$ contains a real line.*
($\mathcal{B}$) *There exists S such that $F = F_{\mathsf{S}}$ and $\mathcal{Z}(F)$ contains no real line.*
($\mathcal{C}$) *F cannot be expressed as F_{S} for a suitable S and $\mathcal{Z}(F)$ contains no real line.*

Moreover if $G \in [F]$ the same statement holds for G.

This gives us a partition of the set of all faces of $\mathrm{P}_{3,4}$ into three parts, which we denote by $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ according to the numeration in corollary 6.7. This partition is compatible with the equivalence relation on $\mathrm{P}_{3,4}$.

In the following sections we will give a complete list of faces of each type. In the last section we will eventually discuss inclusions between faces.

7. Faces in $\mathcal{A}$

Let $F \in \mathcal{A}$ and f be an inner form of F. The next theorem shows that faces in $\mathcal{A}$ are well-understood.

Theorem 7.1 (cf. [Ba], ch. II.12)**.** *Let l be a line in $\mathbb{P}^2(\mathbb{R})$. Then there exists a bijection*

$$\begin{gathered}\left\{F_f\colon\ f \in \mathrm{P}_{3,4},\ \ l^2 \mid f\right\} \longrightarrow \left\{U \subset \mathbb{P}^2(\mathbb{R})\colon\ U \text{ subspace of } \mathbb{P}^2(\mathbb{R})\,\right\}\\ F_f \longmapsto \mathcal{Z}\left(\frac{f}{l^2}\right)\end{gathered}$$

Proof. Since f is nonnegative, f/l^2 is a positive semidefinite quadratic form. Therefore the facial structure of the cone $C := \{f \in \mathrm{P}_{3,4}\colon\ l^2 \mid f\}$ can be identified with the facial structure of symmetric positive definite 3×3–matrices over $\mathbb{R}$. In [Ba] it is shown, that faces of this cone can be parameterized by subspaces of $\mathbb{P}^2(\mathbb{R})$ in terms of simultaneous kernels. □

Therefore each (unordered) pair $(l, U) \in \mathbb{V}_2$, where l is a line, corresponds to a face $F \in \mathcal{A}$. Thus we can regard $\mathcal{A}$ as a subset of $\mathbb{V}_2$. The $\mathcal{G}$–operation on $\mathbb{V}_2$ restricted to $\mathcal{A}$ coincides with above operation of $\mathcal{G}$ on $\mathrm{P}_{3,4}$. Let l, k be distinct real lines and $p, q \in \mathbb{P}^2(\mathbb{R})$ with $p \in l$ and $q \notin l$. Then Lemma 6.4 implies that $\mathcal{A} \subset \mathbb{V}_2$ consists of six equivalence classes, for which representatives are given as follows

$$\begin{aligned} &F_{L_0} := (l, \emptyset),\ F_{L_1} := (l, q),\ F_{L_2} := (l, p)\\ &F_{L_3} := (l, k),\ F_{L_4} := (l, l),\ F_0 = (l, \mathbb{P}^2). \end{aligned}$$

Corollary 7.2. *The set $\mathcal{A}$ splits in six equivalence classes, which are given by F_{L_0}, F_{L_1}, F_{L_2}, F_{L_3}, F_{L_4} and F_0. Every face in $\mathcal{A}$ is exposed. Furthermore we have*

Face F	F_{L_0}	F_{L_1}	F_{L_2}	F_{L_3}	F_{L_4}	F_0
$\dim F$	*6*	*3*	*3*	*1*	*1*	*0*
$\dim J_F$	*3*	*2*	*2*	*1*	*1*	*0*

Proof. We first show the statement concerning given dimensions. Let $F \in \mathcal{A}$ be given by $(l, U) \in \mathcal{A} \subseteq \mathbb{V}_2$. Theorem 7.1 implies that $\dim(F) = \dim\{q \in \mathrm{H}_2\colon\ q(U) = 0\}$, which can be computed easily. Moreover we have $J_S = \{lm\colon\ m \in \mathrm{H}_1, m(U) = 0\}$, which implies $\dim J_S = 2 - \dim U$. It remains

to show that all these faces are exposed. This is clear however, since every face in $\mathcal{A}$ can be written as finite intersection of some F_{p_i} for suitable $p_i \in \mathbb{P}^2(\mathbb{R})$. $\square$

8. Faces in $\mathcal{B}$

For the rest of this paper we set $e_1 = (1\colon 0\colon 0)$, $e_2 := (0\colon 1\colon 0)$, $e_3 = (0\colon 0\colon 1)$ and $e_4 = (1\colon 1\colon 1)$. By definition each face $F \in \mathcal{B}$ is of the form F_{S} for $\mathsf{S} = \{s_1, \dots, s_n, (p_1, l_1), \dots, (p_m, l_m)\}$ and suitable s_i, p_i, l_i. Since $F \notin \mathcal{A}$ we may assume that no three s_i lie on one line and $l_i \cap \{s_1, \dots, s_n, p_1, \dots, p_m\} = \{p_i\}$ for $i = 1, \dots, m$.

Suppose now $2m + n \geq 5$. Let $f \in F_S$ and write $f = q_1^2 + \cdots + q_k^2$. Then $q_1, \dots, q_k$ meet in at least $2m+n$ points (counted with multiplicity). Since $F_S \notin \mathcal{A}$, Bezout's theorem implies that all $q_1, \dots, q_k$ are multiples of an indefinite regular quadratic form q. Since all regular indefinite quadratic forms arise from $g := x^2 - y^2 + z^2$ by coordinate change, we have $F_{\mathsf{S}} \sim F_{q^2}$. We denote F_{q^2} by F_{Q}.

Hence we can assume from now on that $2m + n < 4$. We first consider the case $m = 0$. We set

$$\mathsf{S}_1 := \{e_1\},\ \mathsf{S}_2 := \{e_1, e_2\},\ \mathsf{S}_3 := \{e_1, e_2, e_3\},\ \mathsf{S}_4 := \{e_1, e_2, e_3, e_4\}$$

Lemma 8.1. *For $i = 1, \dots, 4$ the face F_{S_i} is full in I_{S_i}.*

Proof. Using Lemma 3.6 we obtain

$$\begin{aligned}
J_{\mathsf{S}_1} &= \operatorname{span}(y^2, z^2, xy, xz, yz),\\
J_{\mathsf{S}_2} &= \operatorname{span}(z^2, xy, xz, yz),\\
J_{\mathsf{S}_3} &= \operatorname{span}(xy, xz, yz),\\
J_{\mathsf{S}_4} &= \operatorname{span}(xy - yz, yz - xz),
\end{aligned}$$

which makes apparent that conditions (i) and (ii) of theorem 4.4 are satisfied. $\square$

Note that lemma 6.4 implies that each $\mathsf{S} = \{s_1, \dots, s_n\}$ with $n \leq 4$ and no three s_i on a line is $\mathcal{G}$–equivalent to S_n.

Next we consider the case $1 \leq m \leq 2$. We set

$$\mathsf{T}_1 := \{(e_1, e_1 \vee e_3)\}$$
$$\mathsf{T}_2 := \{e_2, (e_1, e_1 \vee e_3)\}$$
$$\mathsf{T}_3 := \{(e_1, e_1 \vee e_3), (e_2, e_2 \vee e_3)\}$$
$$\mathsf{T}_4 := \{e_2, e_3, (e_1 \vee e_4)\}$$

Lemma 8.2. *For $i = 1, \ldots, 4$ the face F_{T_i} is full in I_{T_i}.*

Proof. We compute

$$J_{\mathsf{T}_1} = \operatorname{span}(y^2, z^2, xy, yz)$$
$$J_{\mathsf{T}_2} = \operatorname{span}(z^2, xy, yz)$$
$$J_{\mathsf{T}_3} = \operatorname{span}(z^2, xy)$$
$$J_{\mathsf{T}_4} = \operatorname{span}(yz, x(y - z)).$$

It is enough to check conditions (i)-(iv) of Theorem 4.4. For instance consider T_2. We have $\mathcal{Z}(z^2, xy, yz) = \{e_1, e_2\}$. Moreover $\nabla(xy)(e_2)$ and $\nabla(yz)(e_2)$ are linearly independent. Thus conditions (i) and (ii) are satisfied. Since $\nabla(xy)(e_1) \neq 0$ also condition (iii) is satisfied. For condition (iv) consider the form z^2, which satisfies $\mathcal{Z}(z^2) \cap (e_1 \vee e_3) = \{e_1\}$. For the remaining cases one can argue similarly. □

We summarize the last results in following theorem

Theorem 8.3. *The set $\mathcal{B}$ splits in 10 equivalence classes with respective dimension as follows*

Face F	$F_\emptyset$	F_{S_1}	F_{S_2}	F_{S_3}	F_{S_4}	F_{T_1}	F_{T_2}	F_{T_3}	F_{T_4}	F_{Q}
$\dim F$	*15*	*12*	*9*	*6*	*3*	*9*	*6*	*3*	*3*	*1*
$\dim J_F$	*6*	*5*	*4*	*3*	*2*	*4*	*3*	*2*	*2*	*1*

Furthermore $F_{\mathsf{S}_1}, F_{\mathsf{S}_2}, F_{\mathsf{S}_3}, F_{\mathsf{S}_4}$ and F_{Q} are exposed faces.

Proof. Let $\mathsf{S} = \{s_1, \ldots, s_n, (p_1, l_1), \ldots, (p_m, l_m)\}$ such that $F_{\mathsf{S}} \in \mathcal{B}$. If $n + 2m \geq 5$ then $F_{\mathsf{S}} \sim F_{\mathsf{Q}}$. If $1 \leq n + 2m \leq 4$ then Lemma 6.4 implies the existence of $\sigma \in \mathcal{G}$ and $i \in \{1, 2, 3, 4\}$ such that $F_{\sigma(S)} = F_{S_i}$ or $F_{\sigma(S)} = F_{T_i}$. Thus $\mathcal{B}$ splits in at most 10 equivalence classes. The assertions concerning dimensions are easily verified by computing a basis of I_F and J_F respectively. Since equivalence preserves dimensions of faces, it remains to show that above faces of same dimension are pairwise inequivalent. The form $f := z^4 + y^4 \in F_{\mathsf{T}_1}$ has only one real zero and hence $\sigma(f) \notin F_{\mathsf{S}_2}$ for all $\sigma \in \mathcal{G}$, which implies $F_{\mathsf{S}_2} \not\sim F_{\mathsf{T}_2}$. Similarly

one shows $F_{\mathcal{S}_3} \not\sim F_{\mathcal{T}_2}$ and $F_{\mathcal{S}_4} \not\sim F_{\mathcal{T}_3}, F_{\mathcal{T}_4}$. The form $g := z^4 + x^2y^4 \in F_{\mathcal{T}_3}$ has only 2 real zeros, which implies $F_{\mathcal{T}_4} \not\sim F_{\mathcal{T}_3}$. Lemma 3.4 implies that the faces $F_{\mathcal{S}_i}$ for $i = 1, 2, 3, 4$ are exposed. Since $F_{\mathcal{Q}}$ is equivalent to $F_{\mathcal{S}}$ where $\mathcal{S}$ contains 5 pairwise different real points on an irreducible indefinite conic, also $F_{\mathcal{Q}}$ is exposed. □

9. Faces in $\mathcal{C}$

Last we will construct all remaining faces. Suppose $F \in \mathcal{C}$ and let $f \in F$ an inner form of F. We have seen that if $\mathcal{Z}(f)$ is an infinite set then either $F \sim F_Q$ or $F \in \mathcal{A}$. Hence we may assume that $\mathcal{Z}(f)$ is finite. If $t \in \mathcal{Z}(f)$ then either $\operatorname{ord}_t(f) = 2$ or $\operatorname{ord}_t(f) = 4$.

9.1. **A "degenerate" face.** Suppose $\operatorname{ord}_t(f) = 4$. If f had another real zero s then f would intersect the line $s \vee t$ at least 5 times and hence $(s \vee t) \subseteq \mathcal{Z}(f)$; a contradiction. By coordinate change we may assume that $\mathcal{Z}(f) = \{e_1\}$. Thus the variable x cannot occur in f and therefore $f \in \mathbb{R}[y, z]$. Moreover f has no real zeros on the line $\mathcal{V}(x) \cong \mathbb{P}^1$. Since $f \in F$ is an inner form of F, we must have $\operatorname{ord}_t(g) \geq 4$ for all $g \in F$. Conversely let $g \in \mathrm{H}_4$ such that $\operatorname{ord}_t(g) \geq 4$ (i.e. $g \in \mathbb{R}[y, z]$). Since f is strictly positive on $\mathbb{P}^1(\mathbb{R})$ there exists $\epsilon > 0$ such that $f + \epsilon g$ is nonnegative on the real points of $\mathcal{V}(x)$. Since x does not occur in $f + \epsilon g$, we conclude that $f + \epsilon g$ is globally nonnegative. Thus we have

$$\operatorname{span}(F) = \{g \in \mathrm{H}_4\colon\ \operatorname{ord}_{e_1}(g) \geq 4\} = \mathbb{R}[y, z]_4.$$

We denote the above face by $F_{\mathcal{D}}$ and conclude

$$\dim(F_{\mathcal{D}}) = \dim(\mathbb{R}[y, z]_4) = \binom{5}{4} = 5.$$

9.2. **The remaining faces.** From now on we assume that f has zeros only of order 2. Let $\{s_1, \ldots, s_n\}$ be the set of real zeros of f with $\operatorname{inp}_f(s_i) = \emptyset$ for $i = 1, \ldots, n$ and $p_1, \ldots, p_m$ the remaining real zeros. Let $l_1, \ldots, l_m \subset \mathbb{P}^2$ lines such that $(l_i)_{p_i} \in \operatorname{inp}_f(p_i)$ for $i = 1, \ldots, m$. As usual we write $\mathcal{S} = \{s_1, \ldots, s_n, (p_1, l_1), \ldots, (p_m, l_m)\}$. Since $F \notin \mathcal{B}$ we must have $F \subsetneq F_{\mathcal{S}}$. Instead of viewing at F_f we will work with $J_f = J_F$, which carries all information of F. Since $F \subsetneq F_{\mathcal{S}}$ we have $J_F \subsetneq J_{\mathcal{S}}$. Hence we are interested in vector subspaces $J \subsetneq J_{\mathcal{S}}$ such that J_f for f an inner form of $F \in \mathcal{C}$.

Lemma 9.1. *Let f and $\mathcal{S}$ be as above and $J = J_f$. Then*

(i) $\mathcal{Z}(J) = \{s_1, \ldots, s_n, p_1, \ldots, p_m\}$
(ii) $\dim G_J(s_i) \geq 1$
(iii) $G_J(p_j) \neq \emptyset$

for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Moreover we have $\mathcal{Z}(E_J(p_j)) \cap l_j \neq \{p_j\}$ for at least one $1 \leq j \leq m$.

Proof. Since $\mathcal{Z}(f) = \{s_1, \ldots, s_n, p_1, \ldots, p_m\}$ and $J \subset J_{\mathcal{S}}$ the first assertion must hold. Fix $1 \leq i \leq n$ and choose affine coordinates in s_i. Suppose $G_J(s_i) < 1$, then there exists $v \in \mathbb{R}$ such that $\nabla q(s_i)$ is a multiple of v for each $q \in J$. Hence

$$\widetilde{f}(0, y) = \lambda v^t (1, y)^t$$

for some $\lambda \in \mathbb{R}$. But this polynomial has a real zero (for $v = (1, 0)^t$, this zero is at infinity) and hence $\operatorname{inp}_f(s_i) \neq \emptyset$, which is a contradiction. This proves the second assertion. If $G_J(p_j) = \emptyset$ for a $1 \leq j \leq m$, then every $q \in J$ has a singularity in p_j. Hence $\operatorname{ord}_{p_j}(f) = 4$, which is also a contradiction. Last suppose $\mathcal{Z}(E_J(p_j)) \cap l_j \neq \{p_j\}$ for all j. Then theorem 5.2 implies that $F = F_{\mathcal{S}}$, which is a contradiction. $\square$

Lemma 9.2. *Let $F \subseteq \mathrm{P}_{3,4}$ be a face and let J be an arbitrary subspace of J_F with basis $q_1, \ldots, q_k$. If*

$$q_1^2 + \cdots + q_k^2 - \epsilon h^2$$

is indefinite for all $h \in J_F \setminus J$ and all $\epsilon > 0$. Then $J = J_G$ for a face $G \subseteq F$.

Proof. We set $g := q_1^2 + \cdots + q_k^2$ and claim $J = J_g$. Since $g \in F$ we have $F_g \subseteq F$. By definition we have $q_1, \ldots, q_k \in J_g$ and thus $J \subseteq J_g$. Let now $h \in J_g \subseteq J_F$. Then there exists $\epsilon > 0$ such that $g - \epsilon h^2$ is nonnegative. Thus we must have $h \in J$. $\square$

Corollary 9.3. *Let $F \subseteq \mathrm{P}_{3,4}$ a face and J a subspace of J_F of codimension 1 with basis $q_1, \ldots, q_k$. If there exists $h \in J_F \setminus J$ such that*

$$q_1^2 + \cdots + q_k^2 - \epsilon h^2$$

is indefinite for all $\epsilon > 0$ then there exists a face $G \subsetneq F$ such that $J = J_G$.

Proof. We apply the preceding lemma. We set $g := q_1^2 + \cdots + q_k^2$. Suppose there exists $p \in J_F \setminus J$ and $\epsilon > 0$ such that $g - \epsilon p^2$ is nonnegative. This implies $p \in J_g$. On the other hand we have $J \subseteq J_g$. Since J has codimension 1 in J_F we must have $J_g = J_F$ and in particular $h \in J_g$, which is a contradiction. $\square$

Lemma 9.1 provides a method to find all faces in $\mathcal{C}$. We proceed in the following way. First we fix a configuration of S. Next we will compute all subspaces J of J_{S} which satisfy the conditions of lemma 9.1. Then we will cancel out redundancies regarding equivalent faces. Last we will use Corollary 9.3 in order to show that these subspaces arise in fact from faces. Since there are only 8 inequivalent configurations of S this procedure will eventually give us all faces of $\mathcal{C}$.

We will now distinguish cases according to different types of S. If $m = 0$ then theorem 4.4 implies that f is an inner Form of F_{S} and hence $F = F_{\mathsf{S}} \in \mathcal{B}$. Thus we may assume $m > 0$.

The first interesting case is $m = 1$ and $n = 0$. We may assume $\mathsf{S} = \mathsf{T}_1 = \{(e_1, e_1 \vee e_2)\}$. A basis of J_{T_1} is given by

$$\left\{xy, y^2, yz, z^2\right\}.$$

We are interested in subspaces $J \subset J_S$ satisfying the conditions of lemma 9.1. We write $q_i := A_i xy + B_i y^2 + C_i yz + D_i z^2$ for the basis elements of J, where $1 \le i \le \dim J$. Condition (iii) implies that $A_i \neq 0$ for one i. Therefore we may assume $A_1 = 1$ and $A_i = 0$ for $i \neq 1$. Thus $E = \{q \in J\colon\ \nabla q(e_1) = 0\}$ is spanned by the q_i with $i = 2, \dots, \dim J$. Since we demand $\mathcal{Z}(E) \cap (e_1 \vee e_3) \neq \{e_1\}$ we also have $D_i = 0$ for $i \neq 1$. We distinguish cases according to the dimension of J.

Case 1: $\dim J = 3$.
In this case we may assume that J has basis

$$\{xy + Dz^2, yz, y^2\}, \quad D \in \mathbb{R}.$$

We have $\mathcal{Z}(J) = \{e_1\}$ if and only if $D \neq 0$.

The matrix $M = \mathrm{diag}(\frac{1}{D}, 1, 1)$ induces an element of $\mathrm{stab}(\mathsf{T}_1) \subset \mathcal{G}$, sending J to $\mathrm{span}\{xy + z^2, yz, y^2\}$. Therefore we assume without loss of generality $D = 1$. Last we check the condition of Corollary 9.3. For $\epsilon > 0$ we set

$$g := (xy + z^2)^2 + y^2 z^2 + y^4 - \epsilon z^4.$$

For $t \in \mathbb{R}$ we have

$$g(1, -t^2, t) = t^8 + t^6 - \epsilon t^4,$$

which is negative for all $\epsilon > 0$ and sufficiently small t. For $f = (xy + z^2)^2 + y^2 z^2 + y^4$ we denote F_f and J_f by $F^*_{\mathsf{T}_1}$ and $J^*_{\mathsf{T}_1}$ respectively.

Case 2: $\dim J = 2$
We start with two basis vectors $q_1 = xy + B_1y^2 + C_1yz + D_1z^2$ and $q_2 = B_2y^2 + C_2yz$. Again we must have $D_1 \neq 0$, otherwise q_1 and q_2 vanish on the line $\mathcal{V}(y)$. If $D_1 \neq 0$ the line $\mathcal{V}(y)$ intersects q_1 of order two in e_1. Suppose $C_2 \neq 0$, then q_2 factors in two distinct lines. Since $\mathcal{V}(y)$ intersects q_1 of order two in e_1, the other line has only intersects q_1 of order one in e_1, and therefore there exists another real intersection with q_1, which is a contradiction to $\mathcal{Z}(J) = \{e_1\}$. Thus we must have $C_2 = 0$ and we may assume $q_1 = xz + Cyz + Dz^2$ and $q_2 = y^2$ with $C, D \in \mathbb{R}$, $D \neq 0$.

The matrix
$$M = \begin{pmatrix} 1 & 0 & -C \\ 0 & 1/D & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
induces an element of $\mathrm{stab}(\mathcal{T}_1) \subset \mathcal{G}$ sending J to $\mathrm{span}\{xy + z^2, y^2\}$. Therefore we can assume $D = 1, C = 0$.

By construction J satisfies condition (i),(ii) and (iii) of lemma 9.1. Since $J \subset J^*_{\mathcal{T}_1}$ we can apply Corollary 9.3 and set
$$g := (xy + z^2)^2 + y^4 - \epsilon y^2 z^2, \quad \epsilon > 0.$$
For $t \in \mathbb{R}$ we have
$$g(1, -t^2, t) = t^8 - \epsilon t^6,$$
which is negative for all $\epsilon > 0$ and sufficiently small t. For $f = (xy + z^2)^2 + y^4$ we denote F_f and J_f by $F^{**}_{\mathcal{T}_1}$ and $J^{**}_{\mathcal{T}_1}$ respectively.

Case 3: $\dim J = 1$
In this case J is spanned by a single element q. By lemma 9.1 we must have $\mathcal{Z}(q) = \{e_1\}$ on the other hand we demand $\nabla q(e_1) \neq 0$. Such a $q \in \mathrm{H}_2$ cannot exist.

Thus up to know we have classified all faces F_f which have an inner form f with exactly one real zero.

We will now consider the case $m = 1$ and $n = 1$. It is enough to consider $\mathcal{T}_2 = \{e_2, (e_1, e_1 \vee e_3)\}$. A basis of $J_{\mathcal{T}_2}$ is given by
$$\left\{xy, yz, z^2\right\}.$$
We assume that a subspace $J \subset J_{\mathcal{T}_2}$ has basis $q_i = A_ixy + B_iyz + C_iz^2$ for $1 \leq i \leq \dim J$. We distinguish cases according to the dimension of J.

Case 1: $\dim J = 2$
We first look at e_1. Condition (iii) of lemma 9.1 implies that the monom xy has to occur in one of the q_i. Hence we can assume, that $A_1 = 1$ and $A_2 = 0$. Then $E = \{q \in J\colon\ \nabla q(e_1) = 0\}$ is spanned by q_2. In particular we have $\mathcal{Z}(E) \supset (e_1 \vee e_2)$, thus the last condition holds independently of the choice of the remaining coefficients. For e_2 condition (ii) of lemma 9.1 implies $B_2 \neq 0$. Thus we can assume $B_2 = 1$ and $B_1 = 0$. Last we consider $\mathcal{Z}(J) = \mathcal{Z}(xy + C_1z^2, yz + C_2z^2)$. The form q_2 factors in the linear forms $l_1 = z$ and $l_2 = y + C_2z$. Then $e_1, e_2 \in \mathcal{Z}(l_1)$ and $e_1 \in \mathcal{Z}(l_2)$. Since $l_1 \neq l_2$ the forms q_1 and q_2 intersect transversely in e_2. Since we demand $\mathcal{Z}(q_1, q_2) = \{e_1, e_2\}$ the forms q_1 and q_2 have to be coprime and must intersect of order 3 in e_1. Thus l_2 must intersect q_2 of order 2 in e_1, which implies $C_2 = 0$. On the other hand we must demand $C_1 \neq 0$ otherwise $y \mid q_1$ and $y \mid q_2$. Thus we can assume that a basis of J has the form $q_1 = xy + Cz^2$, $q_2 = yz$ with $C \neq 0$. Applying the projective transformation induced by the matrix $\operatorname{diag}(1, C, 1)$ we can assume that J has basis $\{xy + z^2, yz\}$. By construction J satisfies condition (i)-(iii) of lemma 9.1. We apply Corollary 9.3 and set $f := (xy + z^2)^2 + y^2z^2$. Since J has codimension 1 in $J_{\mathcal{T}_2}$ it is enough to show that there exists no $\epsilon > 0$ such that $g := f - \epsilon z^4$ is nonnegative. In fact we have

$$g(1, -t^2, t) = t^6 - \epsilon t^4,$$

which is negative for all $\epsilon > 0$ and sufficiently small $t \in \mathbb{R}$. For $f = (xy + z^2)^2 + y^2z^2$ we denote F_f and J_F by $F^*_{\mathcal{T}_2}$ and $J^*_{\mathcal{T}_2}$ respectively.

Case 2: $\dim J = 1$.
In this case we can assume that J is spanned by a single form p. But for each $p \in \mathrm{H}_2$ we have $|\mathcal{Z}(p)| \leq 1$ or $|\mathcal{Z}(p)| = \infty$. Thus J does not meet the requirements of lemma 9.1.

Since $\dim J_{\mathcal{T}_3} = \dim J_{\mathcal{T}_3} = 2$ the preceding argument shows also that there exists no face $F \in \mathcal{C}$ with $F \subset F_{\mathcal{T}_3}$ or $F \subset F_{\mathcal{T}_4}$. In particular we have now obtained a complete description of $\mathcal{C}$. The following theorem summarizes the results of this section.

Theorem 9.4. *The set $\mathcal{C}$ splits in 4 equivalence classes with respective dimensions as follows*

Face F	$F_{\mathcal{D}}$	$F^*_{\mathcal{T}_1}$	$F^{**}_{\mathcal{T}_1}$	$F^*_{\mathcal{T}_3}$
$\dim F$	*5*	*6*	*3*	*3*
$\dim J_F$	*3*	*3*	*2*	*2*

Proof. The dimensions in the last line were obtained during the construction of the according faces. To obtain the dimension of F we use the fact $J_F^{[2]} = \operatorname{span}(F)$. It is left to show that these faces are pairwise inequivalent. However this is apparent because the $\mathcal{G}$–operation preserves the dimension and the number of common real zeros of a face. $\square$

Remark 9.5. One can compute the entire list of faces of $\mathrm{P}_{3,4}$ without assuming that a nonnegative form is a sum of squares. Since all extremal forms are perfect squares and every element in a cone is a finite sum of extremal elements, one obtains an alternative proof that every nonnegative ternary quartic is a sum of squares.

Now we have an entire list of faces of $\mathrm{P}_{3,4}$. In the final two sections we will discuss exposednes of faces and inclusions between them.

10. Exposed faces

We have already seen that all faces in $\mathcal{A}$ and faces which are equivalent to one of $F_\emptyset$, F_{S_1}, F_{S_2}, F_{S_3} ,F_{S_4} or F_{Q} are exposed. We will now show that all remaining faces are not exposed.

Each linear map $L\colon \mathrm{H}_{3,4} \longrightarrow \mathbb{R}$ which is nonnegative on $\mathrm{P}_{3,4}$ induces a positive semidefinite symmetric bilinear form $B_L\colon \mathrm{H}_{3,2} \times \mathrm{H}_{3,2} \longrightarrow \mathbb{R}$ given by $B_L(f,g) = L(fg)$. We define $\ker(B_L) := \{f \in \mathrm{H}_{3,2}\colon\ B_L(f,g) = 0,\ \forall g \in \mathrm{H}_{3,2}\}$.

Lemma 10.1. *Let F be an exposed face defined by L. Then we have* $\ker(B_L) = J_F$*.*

Proof. Let $q \in \mathrm{H}_{3,2}$. Since B_L is positive semidefinite we have

$$\begin{aligned} q \in J_F &\iff q^2 \in F \\ &\iff L(q^2) = 0 \\ &\iff B_L(q,q) = 0 \\ &\iff q \in \ker(B_L). \end{aligned}$$

$\square$

Lemma 10.2. *Let $F \subseteq \mathrm{P}_{3,4}$ a face. If there exists $f \in J_F$ and $g, c, d \in \mathrm{H}_{3,2}$ such that $0 \not\equiv c \equiv d \pmod{J_F}$ and $fg = cd$, then F is not exposed. This is in particular the case if we find $f \in J_F$ and $c \notin J_F$ such that $fg = c^2$.*

Proof. Suppose F is exposed and defined by the linear form L. We set $r := c - d \in J_F$. Then we have

$$0 = B_L(f,g) = B_L(c,d) = B_L(d, d+r) = B_L(d,d)$$

which implies $d \in \ker B_L(d,d) = J_F$. This is a contradiction. $\square$

Theorem 10.3. *All faces in $\mathcal{C}$ as well as faces which are equivalent to one of $F_{\mathcal{T}_1}, F_{\mathcal{T}_2}, F_{\mathcal{T}_3}, F_{\mathcal{T}_4}$ are not exposed.*

Proof. We use lemma 10.2. For $i = 1,2,3$ we have $x^2 \in J_{\mathcal{T}_i}$ but $xz \notin J_{\mathcal{T}_i}$. Thus the faces $F_{\mathcal{T}_1}, F_{\mathcal{T}_2}$ and $F_{\mathcal{T}_3}$ cannot be exposed. In order to show that $F_{\mathcal{T}_4}$ is not exposed we note that

$$yzx^2 = \big(z(x-y) + yz + x(y-z)\big) \cdot \big(z(x-y) + yz\big).$$

Since $yz, x(y-z) \in J_{\mathcal{T}_4}$ but $z(x-y) \notin J_{\mathcal{T}_4}$ lemma 10.2 implies that $F_{\mathcal{T}_4}$ is not exposed. In the same way we conclude from $y^2 \in J^*_{\mathcal{T}_1}, J^{**}_{\mathcal{T}_1}$ and $yz \notin J^*_{\mathcal{T}_1}, J^{**}_{\mathcal{T}_1}$ that $F^*_{\mathcal{T}_1}$ and $F^{**}_{\mathcal{T}_1}$ are not exposed. The same argument applied to $y^2 \in J_{\mathcal{D}}$ and $xy \notin J_{\mathcal{D}}$ shows that $F_{\mathcal{D}}$ cannot be exposed. Last we note $0 \not\equiv -xy \equiv z^2 \pmod{J^*_{\mathcal{T}_2}}$ but $yz \in J_{\mathcal{T}_2}$. Thus the equality $-(xz)(yz) = -xyz^2$ implies that $F^*_{\mathcal{T}_2}$ is not exposed. $\square$

11. Inclusions

Definition 11.1. We define a partial ordering $\Subset$ on $\mathcal{F}$ by

$$[F] \Subset [G] \quad \text{if and only if there exists } \sigma \in \mathrm{PGL}_3(\mathbb{R}) \text{ such that } F \subseteq \sigma G.$$

Our goal is to give a complete description of all inclusions between equivalence classes of faces. In our preceding discussion we have chosen all representatives of faces in a way that makes most inclusions (and non-inclusions) obvious. We will discuss the ambiguous cases.

Lemma 11.2. *Let $F, G \subset \mathrm{P}_{3,4}$ be faces with $\dim F = \dim G$. Then we have $F \Subset G$ if and only if $F \sim G$.*

Proof. Suppose $F \subset G$. Since $\dim F = \dim G$ we have $\mathrm{span}(F) = \mathrm{span}(G)$ and thus

$$F = \mathrm{span}(F) \cap \mathrm{P}_{3,4} = \mathrm{span}(G) \cap \mathrm{P}_{3,4} = G.$$

$\square$

To simplify the situation we first discuss the lattice of $\mathcal{A}$ relative to $\Subset$.

Theorem 11.3. *Relative to $\Subset$ the equivalence classes within $\mathcal{A}$ form following lattice.*

$$\begin{tikzcd}
 & F_{\mathsf{L}_0} \arrow[dl, dash] \arrow[dr, dash] & \\
F_{\mathsf{L}_1} \arrow[d, dash] & & F_{\mathsf{L}_2} \arrow[dll, dash] \arrow[d, dash] \\
F_{\mathsf{L}_3} \arrow[dr, dash] & & F_{\mathsf{L}_4} \arrow[dl, dash] \\
 & (0) &
\end{tikzcd}$$

The following theorem clarifies the inclusions between faces in $\mathcal{A}$ and $\mathcal{C}$.

Lemma 11.4. *We have following inclusions*

(i) $F_{\mathsf{L}_2} \Subset F^*_{\mathsf{T}_1}$ *but* $F_{\mathsf{L}_1} \not\Subset F^*_{\mathsf{T}_1}$.
(ii) $F_{\mathsf{L}_4} \Subset F^{**}_{\mathsf{T}_1}$ *but* $F_{\mathsf{L}_3} \not\Subset F^{**}_{\mathsf{T}_1}$.
(iii) $F_{\mathsf{L}_3} \Subset F^*_{\mathsf{T}_2}$ *but* $F_{\mathsf{L}_4} \not\Subset F^*_{\mathsf{T}_2}$.
(iv) $F_{\mathsf{Q}} \Subset F^*_{\mathsf{T}_1}, F^{**}_{\mathsf{T}_1}, F^*_{\mathsf{T}_2}$.

Proof. First we note that for a face $F \in \mathrm{P}_{3,4}$ and $(l,U) \in \mathcal{A}$ we have $(l,U) \subset F$ if and only if F contains a form $l^2 q$ with $q \in \mathrm{P}_{3,4}$ and $\mathcal{Z}(q) = U$.

(i) Since $z^2, yz \in J^*_{\mathsf{T}_1}$ we have $f = z^2(y^2+z^2) \in F^*_{\mathsf{T}_1}$ and thus $F_{\mathsf{L}_2} \Subset F^*_{\mathsf{T}_1}$. Suppose now there exists a line l and q a positive semidefinite quadratic form with $\mathcal{Z}(q) = s \notin \mathcal{Z}(l)$ such that $g = l^2 q \in F^*_{\mathsf{T}_1}$. If $s \neq e_1$ then $g \in F_{\mathsf{T}_1}$ implies that $l = y$. But then we have $F_{\mathsf{L}_0} \Subset F_{f+g}$ which is a contradiction to $F_{f+g} \subset F^*_{\mathsf{T}_1}$ and $\dim F^*_{\mathsf{T}_1} < \dim F_{\mathsf{L}_0}$. Hence we must have $s = e_1$. Hence we must have $q = Ay^2 + Bz^2$ with $A, B > 0$. Since $e_1 \notin \mathcal{Z}(l)$ we obtain $\mathrm{inp}_g(e_1) = \emptyset$, which is a contradiction.

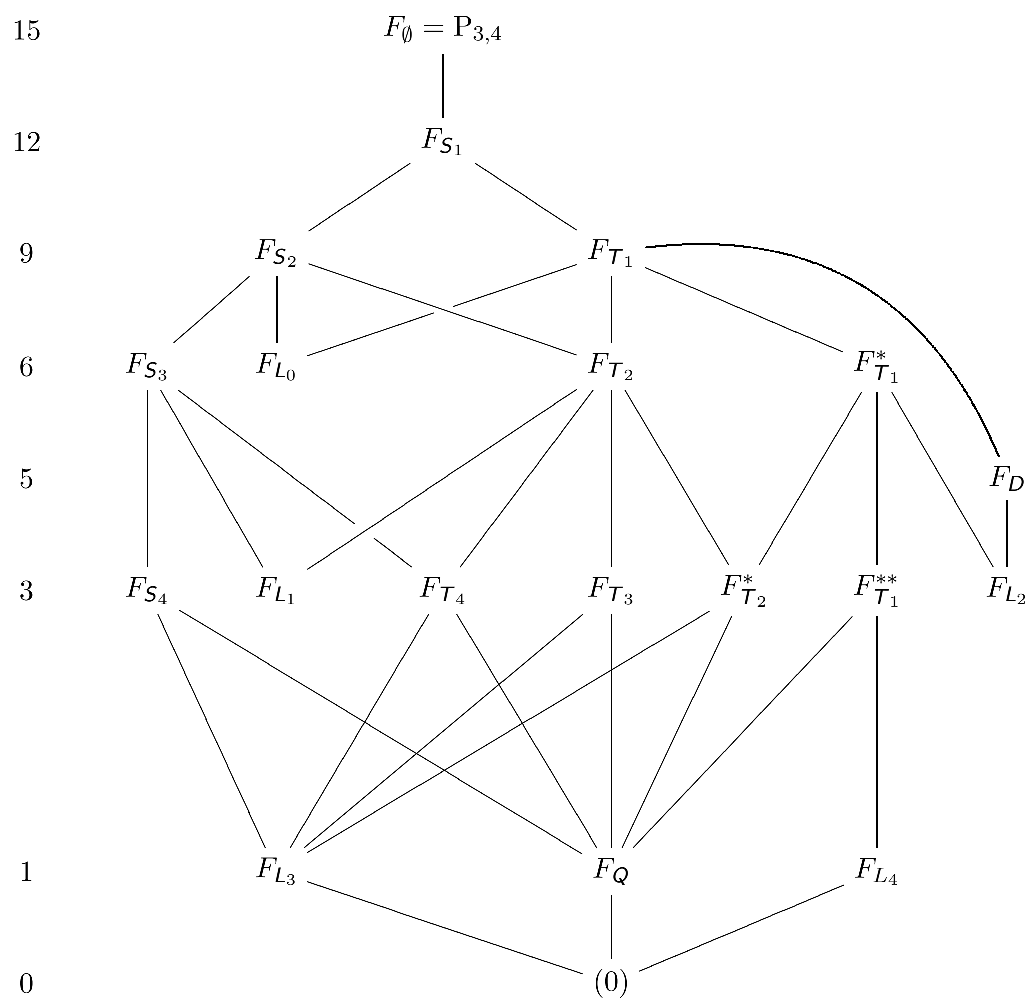
(ii) Since $y^2 \in J^{**}_{\mathsf{T}_1}$ we have $y^4 \in F^{**}_{\mathsf{T}_1}$ and in particular $F_{\mathsf{L}_4} \Subset F^{**}_{\mathsf{T}_1}$. Let l_1, l_2 be two different lines in $\mathbb{P}^2(\mathbb{R})$ with intersection s and set $f = l_1^2 l_2^2$. Suppose $f \in F^{**}_{\mathsf{T}_1}$. If $s \neq e_1$, then l_1 or l_2 must contain $e_1 \vee e_3$. Thus we can assume $l_1 = y$. But then $F_{\mathsf{L}_2} \Subset F_{f+y^4} \subset F^{**}_{\mathsf{T}_1}$. This is a contradiction since $\dim F_{\mathsf{L}_2} = \dim F^{**}_{\mathsf{T}_1}$ but $F^{**}_{\mathsf{T}_1} \neq F_{\mathsf{L}_2}$. Thus we must have $s = e_1$ in particular $\mathrm{ord}_{e_1}(f) = 4$. Recall that $J^{**}_{\mathsf{T}_1}$ has basis $\{y^2, xy + z^2\}$. Since y^2 is the only element having order 2 in e_1, we conclude that y^4 is the only element in $F^{**}_{\mathsf{T}_1}$ with $\mathrm{ord}_{e_1} = 4$. Therefore we have $F_{\mathsf{L}_3} \not\Subset F^{**}_{\mathsf{T}_1}$.

(iii) Since $yz \in J^*_{\mathsf{T}_2}$ we have $F_{\mathsf{L}_3} \Subset F^*_{\mathsf{T}_2}$. Suppose there exists a line l such that $f := l^4 \in F^*_{\mathsf{T}_2}$. Since l has to vanish at e_1 and e_2 we must have $l = z$. But then $z^4 \in F^*_{\mathsf{T}_3}$, which is not the case.

(iv) The form $(xy + z^2)^2$ is a square of a regular indefinite quadratic form contained in $F^*_{\mathsf{T}_1}$,$F^{**}_{\mathsf{T}_1}$ and $F^*_{\mathsf{T}_2}$.

□

Eventually we draw a picture of the lattice of equivalence classes of faces with respective dimensions. To keep the picture clear we omit inclusions within $\mathcal{A}$, which are given by theorem 11.3.



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